

Experimental Verification of Bell's Theorem: Quantum Interference and Entanglement

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Abstract. Bell's Theorem explains an unsolved mystery in quantum mechanics (the EPR paradox) that disproves Local Hidden Variable Theory (LHV). We go over Bell's Inequality that assumes LHV and then conduct a simple experiment that shows Quantum Mechanics violates this inequality, hereby disproving LHV as a valid theory.

Key words: Quantum Entanglement; Bell's Theorem

1. Introduction

Quantum mechanics was incomplete and Einstein knew it – it didn't form a fully comprehensive description of physical reality. He died without resolving this mystery but decades later, John Bell would formulate a theory that explained this puzzle. This experiment explores Bell's Theorem and its description of quantum mechanics.

1.1. Theory

1.1.1 The Einstein Posen Rose Paradox

The Einstein Posen Rose Paradox explains a contradiction of quantum mechanics that describes the theory as incomplete [3]. This theory is a half-finished puzzle. That is to say that there is still a missing variable that the quantum-mechanical wavefunction isn't capturing in physical reality, which for decades was formulated with a Local Hidden Variable Theory (LHV). Later John Bell would disprove LHV.

To grasp the intuition, propose the gedanken experiment proposed by David Bohm: $\pi^0 \rightarrow e^- + e^+$. A neutral pi meson decays into an electron and a positron. The spin configuration is naturally antisymmetric:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

The spin states are entangled in superposition. Naturally upon measuring the spins state in detector A, you'd know the spin state of the other particle. Does this then break relativity where a signal travels faster than speed of light between the two? Additionally, imagine you were traveling close to the speed of light. Depending on the direction, you could witness detector A before detector B or vice versa. How is this instantaneous information possible and resolvable?

Now LHV theory implies that the system specifies a set of instructions for each particle that specifies ahead of time what the results of the measurements will be [5]. Let's assume there is a hidden variable λ following the derivation in Griffiths [4]. We have measurements for the electron A and positron B at arbitrary angles a, b, c :

$$A(a, \lambda), B(b, \lambda) = \pm 1$$

$$E(a, b) = \int A(a, \lambda)B(b, \lambda)\rho(\lambda)d\lambda$$

E takes on the value of the expected value of the correlation. Assuming perfect anti-correlation, the positron and electron will measure different signs by preservation of angular momentum in the neutral pion decay:

$$B(b, \lambda) = -A(b, \lambda)$$

We can use this to rewrite:

$$E(a, b) = - \int A(a, \lambda)A(b, \lambda)\rho(\lambda)d\lambda$$

Most importantly, if LHV was possible and the set of instructions were written ahead of time, then the spins should have been decided for an arbitrary number of angles. Let's introduce angle c .

$$E(a, b) - E(a, c) = - \int [A(a, \lambda)A(b, \lambda) - A(a, \lambda)A(c, \lambda)]\rho(\lambda)d\lambda$$

$$= - \int \rho(\lambda)[1 - A(b, \lambda)A(c, \lambda)]A(a, \lambda)A(b, \lambda)d\lambda \quad \text{where } A(b, \lambda)^2 = 1$$

$$|E(a, b) - E(a, c)| \leq \int \rho(\lambda)[1 - A(b, \lambda)A(c, \lambda)]d\lambda \quad \text{where } |A(a, \lambda)A(b, \lambda)| = 1$$

$$\leq \int \rho(\lambda)d\lambda - \int \rho(\lambda)A(b, \lambda)A(c, \lambda)d\lambda$$

$$\leq 1 + E(b, c)$$

$$|S| \leq 1 + E(b, c)$$

This is the Bell's Inequality set by the condition of LHV. By breaking this inequality we can prove using the contrapositive statements $P \rightarrow Q$ and $\neg Q \rightarrow \neg P$. (LHV \rightarrow Bell's Inequality. Then violating Bell's Inequality \rightarrow no LHV). The idea being that if LHV was real, then the set of instructions known would be known at a very possible angle as well.

1.1.2 Clauser, Horne, Shimony, and Holt's Version

Following from the previous simple thought experiment, let's assume two photons in entangled states (as we have later in our experiment). Exactly like before, we can find a expected value of correlation:

$$E(a, b) = (+1)[P_{VV}(a, b) + P_{HH}(a, b)] + (-1)[P_{VH}(a, b) + P_{HV}(a, b)] = \frac{N_{vv} + N_{hh} - N_{vh} - N_{hv}}{N_{total}} \quad (1)$$

E is the expected value measure of correlation. $|VV\rangle, |HH\rangle$ being +1 correlated and $|VH\rangle, |HV\rangle$ being -1 correlated. So taking the probabilities of these correlated values gives a value $\in [-1, 1]$ that estimates how correlated or anticorrelated or randomized the photon

pairs are.

We can define the following for S:

$$S = E(a, b) - E(a, b') + E(a', b) + E(a', b') \quad (2)$$

$$= - \int \rho(\lambda)[A(a, \lambda)B(b, \lambda) - A(a, \lambda)B(b', \lambda) + A(a', \lambda)B(b, \lambda) + A(a', \lambda)B(b', \lambda)]d\lambda \quad (3)$$

$$s(\lambda) = A(a, \lambda)B(b, \lambda) - A(a, \lambda)B(b', \lambda) + A(a', \lambda)B(b, \lambda) + A(a', \lambda)B(b', \lambda) \quad (4)$$

$$= A(a, \lambda)[B(b, \lambda) - B(b', \lambda)] + A(a', \lambda)[B(b, \lambda) + B(b', \lambda)] \quad (5)$$

$$(6)$$

Since we defined S so, we have a restriction on $|s(\lambda)|$. For both possibilities: $B(b, \lambda) = B(b', \lambda)$ or $B(b, \lambda) = -B(b', \lambda)$ we'd zero out a term and have $s = \pm 2$ yielding the Bell's Inequality:

$$|S| \leq 2 \quad (7)$$

Instead quantum mechanics predicts:

$$|S| \leq 2\sqrt{2}$$

This derivation further explained in Dehlinger&Mitchell [2].

1.1.3 Optics Background

Polarized light Our laser beam is unpolarized so we can assume the photon is some unknown angle relative to your horizontal and vertical states.

$$|\psi\rangle = \cos \phi |H_{las}\rangle + \sin \phi |V_{las}\rangle$$

Half Waveplates Half waveplates work using a birefringence crystal. The material is anisotropic so light passes through differently relative to its the optical axis intrinsic to the material.

$$T_{||} = \cos^2(2\theta), T_{\perp} = \sin^2(2\theta)$$

So we can see that the material acts a polarization rotator [1, s. 1.46]. (As a side note, for circular polarization as opposed to linear we have the sugar water barber pole effect. Similar concept but different material.)

Polarizing Cube Beam Splitters We also use polarizing beam splitters (PBS) which allow one polarization of light to transmit through and the reflects the other at 90 degrees. This allows us to detect photons of a certain set polarization angle aligned with the PBS.

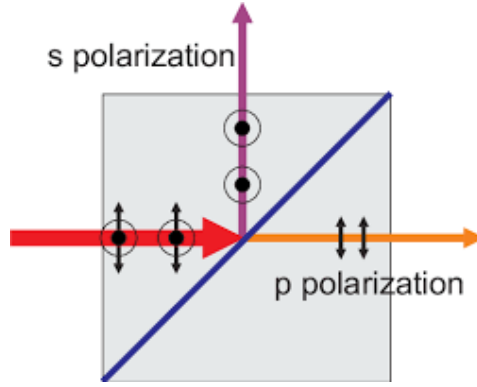


Figure 1: The PBS only transmitting p-polarization

2. Setup

Blue laser of 405 nm wavelength at ≤ 30 mW of energy is pumped through the laser diode and reflected by two right angle mirrors. We use 405 nm as it splits neatly into two 810 nm photons.

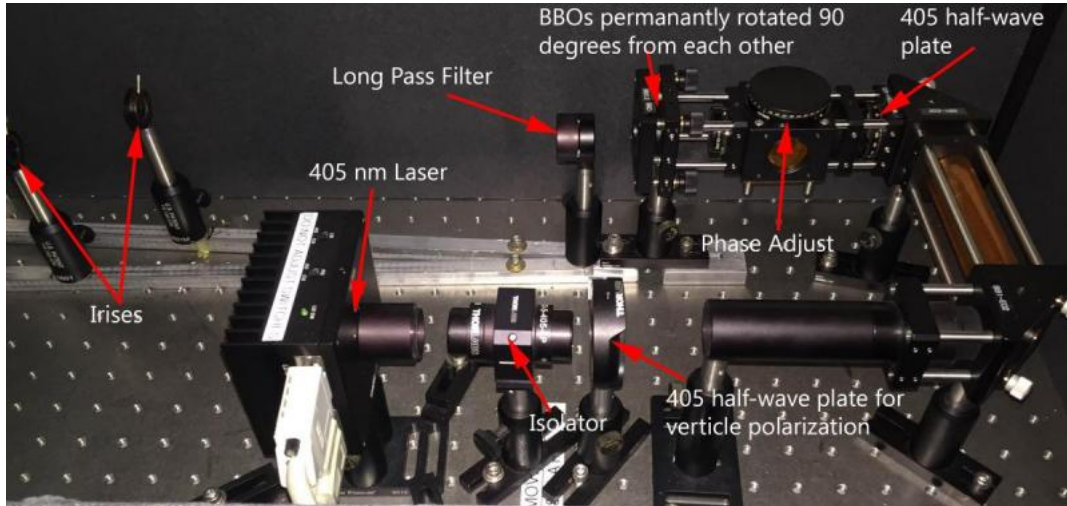


Figure 2: Initial box

The photon in question is in the quantum state (relative to our horizontal and vertical). Since it's unpolarized light, we don't know the polarization angle so it's an arbitrary ϕ .

$$|\psi\rangle = \cos \phi |H_{las}\rangle + \sin \phi |V_{las}\rangle$$

The blue laser goes through a half-wave plate that rotates the light. Say the final rotation by θ_l

$$|\psi\rangle = \cos \theta_l |H_{las}\rangle + \sin \theta_l |V_{las}\rangle, \quad \theta_l = 2\theta - \phi$$

Then we introduce a phase shift using the phase adjust knob to counteract a phase shift presented by the BBOs themselves. From our knowledge of birefringent materials [1, s 1.46] the material introduces a phase lag. We can take the difference and sum it in a single $e^{i\phi}$ phase difference term.

$$|\psi\rangle = \cos \theta_l |H_{las}\rangle + e^{i\phi} \sin \theta_l |V_{las}\rangle$$

Upon passing through the BBO pair, some of the photons pass through the first and others through the second (many through directly). In type-I SPDC, we see the splitting.

$$|H_{las}\rangle \rightarrow |V_1 V_2\rangle, \quad |V_{las}\rangle \rightarrow |H_1 H_2\rangle |\psi\rangle = \cos \theta_l |H_1 H_2\rangle + e^{i\phi} \sin \theta_l |V_1 V_2\rangle \quad (8)$$

We adjust these parameters θ_l and ϕ to reach the perfect Bell State:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|H_1 H_2\rangle + |V_1 V_2\rangle) \quad (9)$$

Under this Bell State, we can most easily detect violation in Bell's Inequality. Recall from the prelab that we can detect the purity of our θ_l by checking (0, 0) and (90, 90) contain equal counts and the purity of ϕ by checking the (45, 45) angle matches closely with (0,0) and (90, 90).

Counts	α	β
$N \cos^2 \theta_l$	0°	0°
$N \sin^2 \theta_l$	45°	45°
$N \left \frac{1}{2}(\cos \theta_l + \sin \theta_l e^{i\phi}) \right ^2$	90°	90°
0	0°	90°

Table 1: Calculated predicted coincident counts by adjusting θ_l and ϕ parameters

Counts	α	β
$N/2$	0°	0°
$N/2$	45°	45°
$N/2$	90°	90°
0	0°	90°

Table 2: The ideal Bell State coincident counts.

Half wave plate again are rotating the polarization angle for the specific photon.

$$|\psi\rangle = \frac{1}{2} \left[(\sin \alpha \sin \beta + \cos \alpha \cos \beta) |V_1 V_2\rangle \right. \\ \left. + (\cos \alpha \sin \beta - \sin \alpha \cos \beta) |H_1 V_2\rangle \right. \\ \left. + (\sin \alpha \cos \beta - \cos \alpha \sin \beta) |V_1 H_2\rangle \right. \\ \left. + (\sin \alpha \sin \beta + \cos \alpha \cos \beta) |H_1 H_2\rangle \right]$$

Our PBS allow only 'vertical' polarization to pass through. Hence capturing the $|VV\rangle$ or $|HH\rangle$ etc by varying the half-wave plates. Long pass filters just remove unwanted wavelengths. Detectors lens narrow the light beam into the fiber optic cable which passes to the ADP for photon counting.

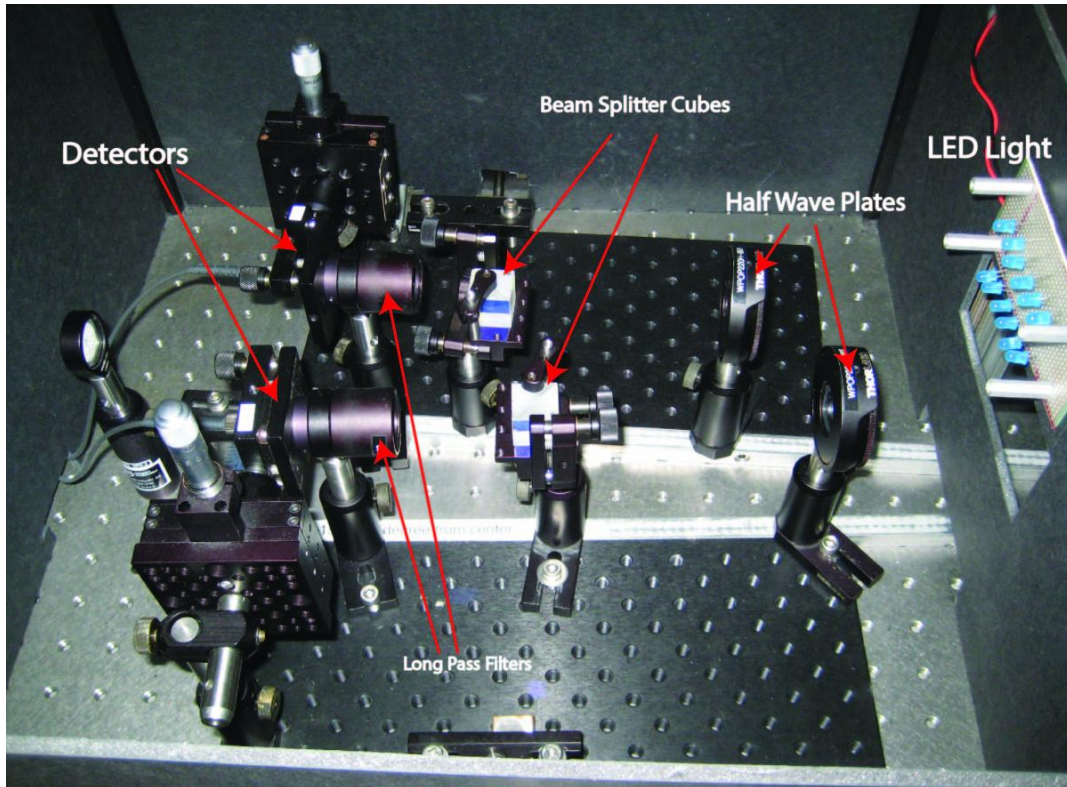


Figure 3: Initial box

3. Procedure and Analysis

Alignment First and foremost, we had to make sure our blue-violet laser beam path was aligned. Since our fiber has a core of 1 micrometer, we need accuracy. We'll remove the lens's protective cover and the half waveplates and shoot a low energy infrared laser from the fiber optic cables through the detectors and backwards through our beam paths. We need the infrared paths to converge onto a single point on the BBOs so using a thin piece of paper we adjust the detector ($xy \theta$) to match the violet dot (set to a very low wattage) exiting the BBOs.

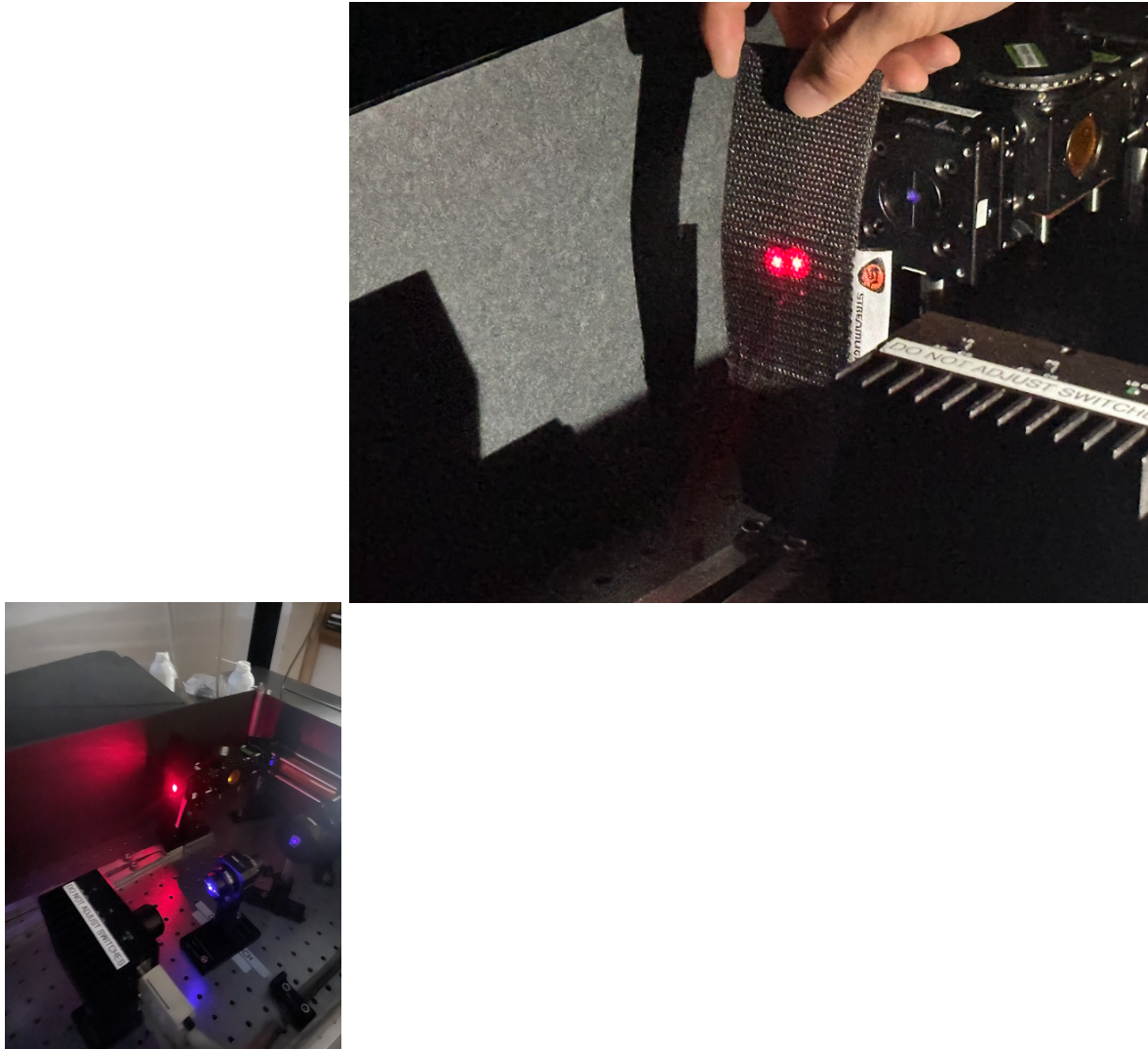


Figure 4: Process of alignment by shooting infrared laser through both detector arms towards a singular spot on the BBOs exit pinhole.

We'll align more by adjusting the detector arms themselves. Since the BBOs emit photon pairs in a cone shape, we'll need our detector lens to be perfectly within that cone. By running the LabView program, we adjust this angle until we maximize our counts on each arm A and B'. We've adjusted the cone angle to be approximately 3.796 degrees yielding maximum counts 38770 and 421967 for A and B' respectively.

Optimizing the Bell State Recall in eq. 9 we can achieve the best violation of the bell inequality. However, we'll need to find an arbitrary offset for the half-waveplates to light up with the polarization angle of the PBS. We basically rotated the half-wave plates on each detector arm to maximize the counts on each arm—thereby aligning the polarization of the photons. Hence getting $|VV\rangle$ or $|HH\rangle$. The idea is that the some of the photons in the beam will hit the first BBO and produce say $|HH\rangle$ while some of the photons will hit the second BBO, which is at a perfect 90 degrees and produce $|VV\rangle$ at this surrounding light cone. Most of the violet beam will pass through.

Using this understanding, we'll rotate the halfwave plate to maximize the $|VV\rangle$ state. Rotate each of the half wave plates to maximize the count on that arm. This means the photon's polarization vector is most aligned with the PBS optical axis. For these values we got $A=-2$ and $B'=2$.

Furthering Bell State Purity We can better our Bell State 9 By adjusting the the laser beam before entering the BBO pairs. We have the phase adjust knob ϕ and the half-wave plate θ_l that we can adjust our Bell State with the equation given in eq. 8.

Testing Bell State Purity We can test our Bell State for purity as shown previously in 2. Essentially, we'll attempt to adjust θ_l until we get equal counts for (0, 0) and (90, 90). Then we'll adjust (45, 45) until we get similar counts to (0, 0) and (90, 90). Finally, we'll try to achieve zero counts for (0, 90). (The reason is further explained in the theory section 2 and prelab but is a simple result of the wavefunction probability function). We settled on $\phi = 22.5$, $\theta_l = 44$ and got the following results:

Index	A	B'	AB'	alpha	beta
0	82987.2	71808.2	2862.41	0.0	0.0
1	57062.9	41415.1	1481.57	45.0	45.0
2	17327.9	16973.6	762.36	90.0	90.0
3	83014.9	17049.7	27.15	0.0	90.0

Table 3: Result of testing our Bell State Purity. We achieve a low (0, 90) count but uneven counts for the rest.

As you can see in the table above, our Bell State was not ideal. We are achieving low counts for (0, 90) but noticeable differences in the other counts. We had believed that this was due to Malus's Law which is shows $I \propto I_0 \cos^2 \theta$ which could explain the intensity drop by 1/2 from (0, 0) to the (45, 45) case, etc. Or we perhaps imperfectly set θ_l not perfectly taking into account the polarization rotation of half-wave plates. Despites our fiddling, we could not achieve a better Bell State and decided to move on – this comes to give less than ideal results later on.

Violating the Bell State Following the dervation described in the theory section 1, we'll simply take the measurements for a, a', b, b' and all their perpendicular values, giving 16 values in total. We have $a = -45, a' = 0$ and $b = -22.5, b' = 22.5$. We record the coincidence counts and all the counts on each detector arm.

Index	A	B'	AB'	alpha	beta
0	48633.5	67458.3	1507.79	-45.0	-22.5
1	48638.4	60670.8	737.79	-45.0	22.5
2	48636.2	17735.2	294.07	-45.0	67.5
3	48664.4	27652.1	971.94	-45.0	112.5
4	82448.6	27633.5	584.42	0.0	112.5
5	82437.0	17348.3	66.20	0.0	67.5
6	82545.2	60675.5	2329.82	0.0	22.5
7	82510.5	67633.1	2669.70	0.0	-22.5
8	54558.3	67494.4	1298.88	45.0	-22.5
9	54602.3	61141.7	1898.92	45.0	22.5
10	54656.5	47014.9	571.10	45.0	67.5
11	54623.7	27851.9	181.59	45.0	112.5
12	16967.0	27885.9	577.37	90.0	112.5
13	16831.9	17268.3	793.49	90.0	67.5
14	16905.7	60748.7	231.43	90.0	22.5
15	16865.8	67321.1	63.15	90.0	-22.5

Figure 5: A table of all the the coincidence counts AB' and the counts for A and B' for all 16 combinations of alpha and beta

We then use the following data and graph them on top of its best fit curve for the predicted function. These are plotting by fixing the alpha term and varying the beta angle on the x-axis by the coincidence counts on the y-axis.

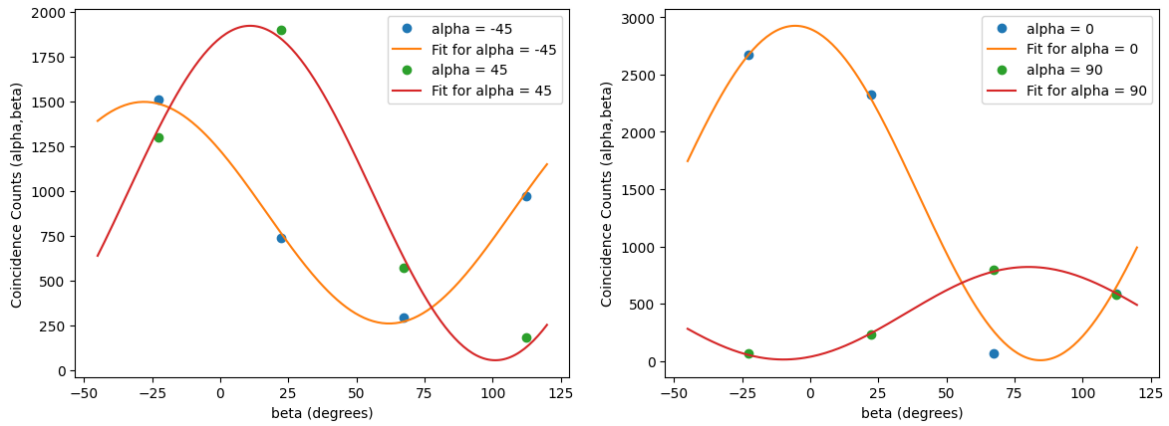


Figure 6: Plotting our data above and its best-fit curve

As you can see, we have a noticeable sinusoidal wave but with certain phase delay and amplitude changes. Of course, this was due to our impure Bell State which we had trouble fixing.

Using the equations we discussed in previous sections 1 and 6, we can calculate our value for $|S|$. Again, the reason can be found in the theory section but the idea being that this particular calculation of S should obey $|S| \leq 2$ assuming local hidden variable (LHV) theory is correct. However violating this ‘Bell’s Inequality’, by contrapositive argument, would mean LHV is incorrect in Quantum Mechanics. Detailed calculations/code for S found in appendix A. We are achieving:

$$S = 2.1369 \tag{10}$$

We've violated Bell's Inequality 7! $|S| > 2$. Quantum Mechanics allows values up to $|S| \leq 2\sqrt{2}$ as discussed in Dehlinger and Mitchell [2].

Comparison with paper's experiment We can compare our results with the experiment from Dehlinger and Mitchell [2].

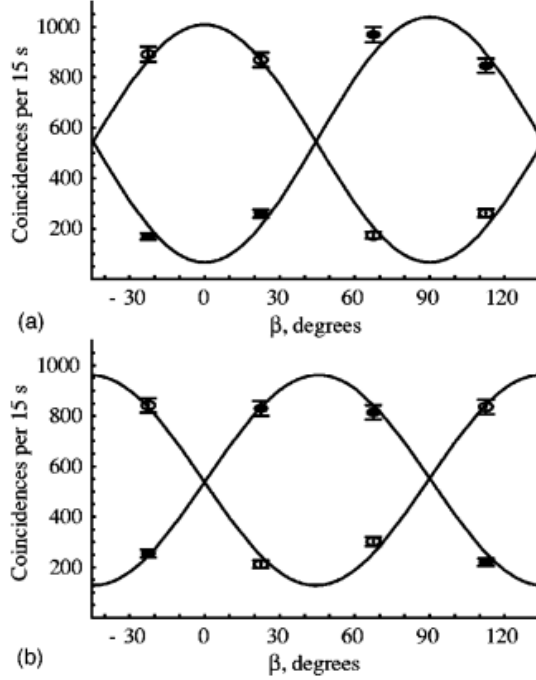


Fig. 6. Typical coincidence counts for the Bell inequality test. (a) Open and closed circles show $\alpha=0^\circ, 90^\circ$, respectively. (b) Open and closed circles show $\alpha=45^\circ, 135^\circ$, respectively. Error bars indicate plus or minus one standard deviation statistical uncertainty. The curves are a fit to Eq. (12).

Figure 7: Their data for similar alpha, beta terms and the best-fit curve plots. They achieve a symmetrical curves all-around.

Their paper did achieve a result of $S = 2.307 \pm 0.035$ which violates the Bell's Inequality. Of course, their Bell State being much more pure and accurate. QM allows values up to 2.828.

What really happens Our current understanding is that information doesn't travel between the photons. Instead, there exists only a correlation between the datasets when you measure both detectors. This means, if you read detector A, you don't actually know B. There is a black box around B. It is only after reading the measurements of detector B that you find a correlation in the data.

4. Conclusion

Through our experimentation we show that a violation of Bell's Inequality is presented, hereby showing the assumption of LHV is false. Our experiment of shooting photons onto detectors and adjustment of the different polarization angles shows that quantum spookiness is much more slippery and non-intuitive than previously encountered.

References

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A. Code Calculations

Found here [final.ipynb](#)

B. Data Set

Found here [data.csv](#)